<u>Note:</u> If G acts on a set S, we often assume that ① $\forall g_1, g_2 \in G$, $x \in S$, $g_1 \cdot (g_2 \cdot s) = (g_1 g_2) \cdot s$ Definition

② If $e \in G$ is the identity, then $e \cdot \cancel{x} = x \quad \forall x \in S$

(often 2nd part of defn – we'll assume this unless specified otherwise.)

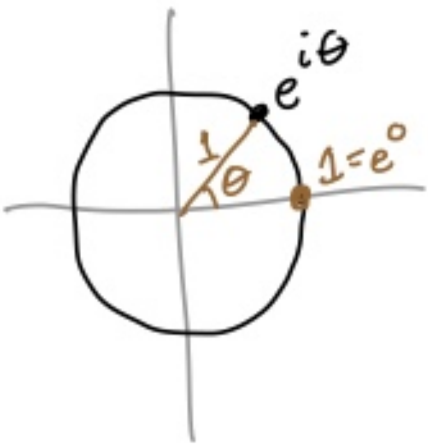
More Examples

$e^{i\theta} = \cos\theta + i\sin\theta = (\cos\theta, \sin\theta)$

① $S^1 = \{e^{i\theta} : \theta \in \mathbb{R}/2\pi\mathbb{Z}\}$ is the unit circle in $\mathbb{C} = \mathbb{R}^2$.

$\{r + 2\pi\mathbb{Z} : r \in \mathbb{R}\}$
$\cong$
$[0, 2\pi]/_{0=2\pi}$

This is a group:



eg $e^{i\frac{\pi}{2}} \cdot e^{i\frac{7\pi}{8}} = e^{i\left(\frac{\pi}{2} + \frac{7\pi}{8}\right)} = e^{i\frac{11\pi}{8}}$

$e^{i\frac{7\pi}{8}} e^{i\frac{7\pi}{8}} = e^{i\frac{14\pi}{8}} = e^{i\frac{7\pi}{4}}$

$e^{i\frac{7\pi}{4}} e^{i\frac{7\pi}{4}} = e^{i\frac{7\pi}{2}} = e^{i\frac{3\pi}{2}}$

S^1 acts on $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$
$= \{(\cos\alpha \sin\phi, \sin\alpha \sin\phi, \cos\phi) : \alpha \in \mathbb{R}/2\pi\mathbb{Z}, \phi \in [0, \pi]\}$.

by

$e^{i\theta}(x, y, z) = ((\cos\theta)x - (\sin\theta)y, (\sin\theta)x + (\cos\theta)y, z)$

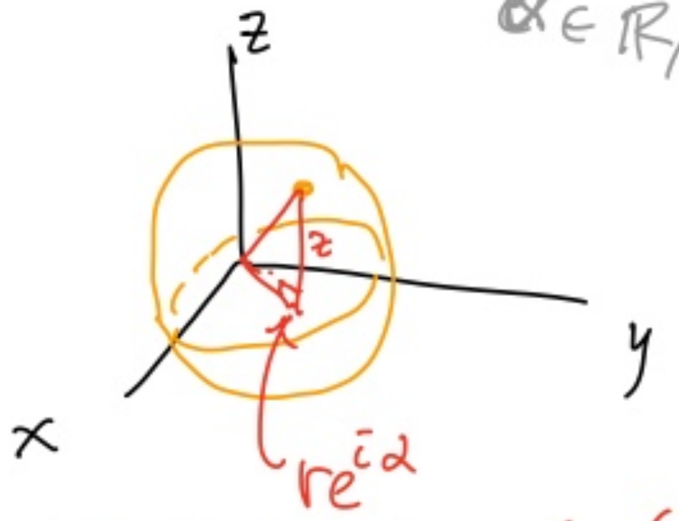
$$\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}\cos\theta x - \sin\theta y\\ \sin\theta + \cos\theta y\end{pmatrix}$$

Better notation: use cylindrical coordinates for S^2

$$S^2 = \{(re^{i\alpha}, z) : r^2 + z^2 = 1\}$$

$z \in \mathbb{R}$, $r \geq 0$

$\alpha \in \mathbb{R}/2\pi\mathbb{Z}$



$$e^{i\theta}(re^{i\alpha}, z) = (re^{i(\alpha+\theta)}, z).$$

$\therefore$ action by $\theta \in S^1$ on S^2 is rotation by θ in the xy plane.

Observe
$$e^{i\theta_1}\left(e^{i\theta_2}(re^{i\alpha}, z)\right) = e^{i(\theta_1+\theta_2)}(re^{i\alpha}, z)$$
$$= \left(e^{i\theta_1}e^{i\theta_2}\right)(re^{i\alpha}, z).$$

$\therefore$ it is a group action.

$1 = e^{i0}$ $\quad 1.(re^{i\alpha}, z) = (re^{i\alpha}, z).$ ✓

Notation with Group Actions:

If the group G acts on the set S, for $x \in S$, the orbit of x (denoted Gx) is $Gx = \{g \cdot x \mid g \in G\} \subseteq S$.

Given $x \in S$, the isotropy subgroup (or stabilizer subgroup) at x is

$$G_x = \{h \in G : h \cdot x = x\}$$

Back to S^1 acting on S^2.

If $(x,y,z) \in S^2$, what is its orbit?

$$S^1(x_0,y_0,z_0) = \{e^{i\theta}(x_0,y_0,z_0) : \theta \in \mathbb{R}/2\pi\mathbb{Z}\}.$$

$$= \{e^{i\theta}(r_0 e^{i\alpha_0}, z_0) : \theta \in \mathbb{R}/2\pi\mathbb{Z}\}$$

$$= \{(r e^{i(\alpha_0+\theta)}, z_0) : \theta \in \mathbb{R}/2\pi\mathbb{Z}\}.$$

$$= \left\{(x,y,z) : z = z_0,\ x^2+y^2 = x_0^2+y_0^2 = 1-z_0^2\right\}$$

$\Rightarrow$ There are two kinds of orbits.

$NP = \{(0,0,1)\}$

$SP = \{(0,0,-1)\}$

if $|z| \neq 1$, circle as orbit.

stabilizer $S^1_{(x_0,y_0,z_0)} = \{ e^{i\theta} \in S^1 : e^{i\theta}(x_0,y_0,z_0) = (x_0,y_0,z_0) \}$
$\subseteq S^1$

$\Rightarrow \quad S^1_{(0,0,1)} = S^1$

$S^1_{(0,0,-1)} = S^1$

if $|z| \neq 1$ $\quad S^1_{(x_0,y_0,z_0)} = \{ \underset{\substack{\uparrow \\ e^{i\theta}}}{1} \} \subseteq S^1$

---

(2) S^1 acts on $S^3 = \{ (z_1, z_2) : |z_1|^2 + |z_2|^2 = 1 \}$

by

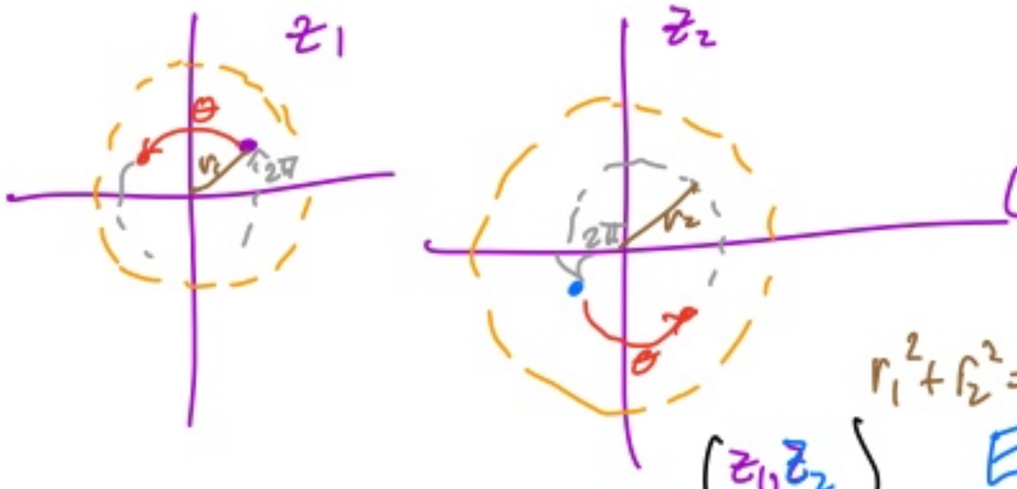
$$e^{i\theta}(z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2).$$

Observe
yes it's a group action

$$e^{i\theta_1}(e^{i\theta_2}(z_1,z_2)) = \ldots = (e^{i\theta_1} e^{i\theta_2})(z_1,z_2)$$

$$1(z_1,z_2) = (1z_1, 1z_2) = (z_1,z_2). \checkmark$$

every orbit is a circle!



z_1 & z_2 can't both be zero → No fixed points.

$r_1^2 + r_2^2 = 1$

(z_1, z_2)

Every stabilizer is the trivial group
Every orbit is a circle.

Cool Fact: S^3 is actually a group!

group of unit quaternions!

$$\begin{array}{ccc} S^1 & \hookrightarrow & S^3 \\ & & \Big\downarrow \text{map each circle to a point.} \\ & & S^2 \end{array}$$

(Hopf Fibration)

S^2 ← 2-d sphere